Quantum Mechanics as a Classical Theory VIII: Second Quantization

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Abstract

We continue in this paper our program of rederiving all quantum mechanical formalism from the classical one. We now turn our attention to the derivation of the second quantized equations, both for integral and half-integral spins. We then show that all the quantum results may be derived using our approach and also show the interpretation suggested by this derivation. This paper may be considered as a first approach to the study of the quantum field theory beginning by the same classical ideas we are supporting since the first paper of this series.

1 Introduction

In this paper we are interested in showing that the concept of second quantization might be built within our purely classical reconstruction of quantum mechanics. More than that, when deriving the relevant results, we are led to a much clearer picture of the 'creation' and 'annihilation' operators action.

We will develop our study in the realm of two important systems where second quantization methods may be applied—which are fermionic and bosonic systems. We show that all the mathematical results of ordinary quantum mechanics may be retrieved by our methods.

We begin the next section obtaining the quantum Schrödinger equation for the harmonic oscillator problem using the *classical* Hamilton's equations together with Liouville's equation. This will be attained with the use of the Wigner-Moyal Infinitesimal Transformation[1, 2, 3]. Then we perform a canonical transformation in the classical phase space to get a new Hamiltonian and, using again the Wigner-Moyal Infinitesimal Transformation, we derive the canonically transformed Schrödinger equation. We then show that this equation is precisely the 'second quantized' one. This result shows that, with our method,

it is possible to quantize physical systems in any generalized phase-space—we left, however, the development of this result to a future paper.

The third section is devoted to the study of classical second quantization of fermionic systems. The same procedures used for bosons will be applied for fermions and we will retrieve again the formalism encountered in the literature.

In the last section we make our final conclusions.

2 Harmonic Oscillator

We begin with the *classical* harmonic oscillator defined by the hamiltonian

$$H = \frac{1}{2m}p^2 + \frac{m\omega^2}{2}q^2 \tag{1}$$

and the *classical* statistical Liouville equation

$$\frac{dF}{dt} = \frac{\partial F}{\partial t} + \frac{dq}{dt}\frac{\partial F}{\partial q} + \frac{dp}{dt}\frac{\partial F}{\partial p} = 0. \tag{2}$$

Using Hamilton's equations

$$\frac{dq}{dt} = \frac{\partial H}{\partial p} = \frac{p}{m} \; ; \; \frac{dp}{dt} = -\frac{\partial H}{\partial q} = -m\omega^2 q \tag{3}$$

we rewrite (2) as

$$\frac{\partial F}{\partial t} + \frac{p}{m}\frac{\partial F}{\partial q} - m\omega^2 q \, \frac{\partial F}{\partial p} = 0. \tag{4}$$

We now define the density function as the Infinitesimal Wigner-Moyal transformation

$$\rho\left(q - \frac{\delta q}{2}, q + \frac{\delta q}{2}; t\right) = \int F\left(q, p; t\right) e^{\frac{i}{\hbar}p\delta q} dp \tag{5}$$

and apply it upon equation (4) to get

$$-i\hbar\frac{\partial \rho}{\partial t} - \frac{\hbar^2}{m}\frac{\partial^2 \rho}{\partial q \partial(\delta q)} + m\omega^2 q \delta q \rho = 0 \tag{6}$$

We now suppose that this density function might be written as

$$\rho\left(q - \frac{\delta q}{2}, q + \frac{\delta q}{2}; t\right) = \Phi^{\dagger}\left(q - \frac{\delta q}{2}; t\right)\Phi\left(q + \frac{\delta q}{2}; t\right) \tag{7}$$

and use the fact that the function $\Phi(q;t)$ is complex and might be written as

$$\Phi\left(q;t\right) = R\left(q;t\right) e^{\frac{i}{\hbar}S(q;t)} \tag{8}$$

where R and S are real functions.

It is then possible to take expression (7)—with (8) for the functions Φ—into expression (6) and, retaining only terms in the zeroth and first order on the infinitesimal parameter δq, to derive the following equation

$$\left[\frac{\partial(R^2)}{\partial t} + \frac{\partial}{\partial q}\left(R^2 \frac{\partial S/\partial q}{m}\right)\right] +$$
$$+ R^2 \frac{i\delta q}{\hbar}\frac{\partial}{\partial q}\left[\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial q^2} + \frac{m\omega^2}{2}q^2\right] = 0. \tag{9}$$

The real and imaginary terms have to be each identically zero and we get the two equations

$$\frac{\partial(R^2)}{\partial t} + \frac{\partial}{\partial q}\left(R^2 \frac{\partial S/\partial q}{m}\right) = 0 \tag{10}$$

and

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 - \frac{\hbar^2}{2mR}\frac{\partial^2 R}{\partial q^2} + \frac{m\omega^2}{2}q^2 = 0 \tag{11}$$

where in this last equation we made the arbitrary constant equal to zero like in our other papers[1].

We may show by straightforward calculations that these equations are precisely those we obtain when we substitute expression (8) in the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \Phi}{\partial q^2} + \frac{1}{2}m\omega^2 q^2 \Phi = i\hbar\frac{\partial \Phi}{\partial t} \tag{12}$$

where now we immediately identify expression (10) as the continuity equation. We then conclude that these two sets of equations (10,11) and (12) have the same mathematical content and so use them interchangeably. Therefore, we have shown that it is possible to derive the harmonic oscillator quantum equation from classical mechanics by means of the Infinitesimal Wigner-Moyal Transformation.

We now make a canonical transformation in the *classical* phase space defined by

$$q_1 = \left[\frac{p}{\sqrt{2\hbar m\omega}} + i\sqrt{\frac{m\omega}{2\hbar}}q\right] \; ; \; p_1 = \left[\frac{p}{\sqrt{2\hbar m\omega}} - i\sqrt{\frac{m\omega}{2\hbar}}q\right] \tag{13}$$

to get the new hamiltonian

$$H_1 = (\hbar\omega)\, q_1 p_1 \tag{14}$$

and the new Liouville equation (F is now a function of q_1 and p_1)

$$\frac{\partial F}{\partial t} + \hbar\omega q_1 \frac{\partial F}{\partial q_1} - \hbar\omega p_1 \frac{\partial F}{\partial p_1} = 0, \tag{15}$$

where we used Hamilton's equations

$$\frac{dq_1}{dt} = \hbar\omega q_1 \; ; \; \frac{dp_1}{dt} = -\hbar\omega p_1. \tag{16}$$

With the definition

$$\rho\left(q_1-\frac{\delta q_1}{2},q_1+\frac{\delta q_1}{2};t\right)=\int F\left(q_1,p_1;t\right)e^{\frac{i}{\hbar}p_1\delta q_1}dp_1 \tag{17}$$

for the density function we follow the same steps as above to get the equation

$$\frac{\partial\rho}{\partial t}+\hbar\omega q_1\frac{\partial\rho}{\partial q_1}-\hbar\omega\frac{\partial}{\partial(\delta q_1)}\left(\delta q_1\rho\right)=0. \tag{18}$$

Imposing the format

$$\rho\left(q_1-\frac{\delta q_1}{2},q_1+\frac{\delta q_1}{2};t\right)=\Phi^\dagger\left(q_1-\frac{\delta q_1}{2};t\right)\Phi\left(q_1+\frac{\delta q_1}{2};t\right) \tag{19}$$

upon the density function and using the expression

$$\Phi\left(q_1;t\right)=R\left(q_1;t\right)e^{\frac{i}{\hbar}S(q_1;t)} \tag{20}$$

we get, with the same considerations as above, the following pair of equations

$$\frac{\partial(R^2)}{\partial t}+\hbar\omega q_1\frac{\partial(R^2)}{\partial q_1}+\hbar\omega R^2=0 \tag{21}$$

and

$$\frac{\partial}{\partial q_1}\left[\frac{\partial S}{\partial t}+\hbar\omega q_1\frac{\partial S}{\partial q_1}\right]=0. \tag{22}$$

The pair of equations (21,22) is equivalent—in the sense defined above—to the Schrödinger equation

$$\hbar\omega\left[-i\hbar q_1\frac{\partial\Phi}{\partial q_1}+\frac{i\hbar}{2}\Phi\right]=i\hbar\frac{\partial\Phi}{\partial t} \tag{23}$$

which can be rewritten as

$$\hbar\omega\left(a^\dagger a+\frac{[a,a^\dagger]}{2}\right)\Phi=i\hbar\frac{\partial\Phi}{\partial t}, \tag{24}$$

if we make the identification

$$a^\dagger=q_1\ ;\ a=-i\hbar\frac{\partial}{\partial q_1}. \tag{25}$$

Equation (24) is nothing but the second quantized equation for the harmonic oscillator when we make the canonical transformation (13) in the operator space and apply it to the Schrödinger equation (12). These calculations show that we may perform the canonical transformation in the classical (functions) or

quantum (operators) phase space as we wish, since the results will be exactly the same.

Equation (23) is a differential equation and we might solve it to obtain the amplitude Φ as

$$\Phi(q_1;t) = q_1^n e^{-iEt/\hbar}\Phi_0, \tag{26}$$

where Φ_0 is a constant usually called the vacuum state. The energy may also be obtained and we get

$$E = \hbar\omega\left(n + \frac{1}{2}\right) \tag{27}$$

as expected.

In the language of operators a and $a^\dagger$, the amplitude in (26) may be written as

$$|\Phi(q_1)\rangle = \left(a^\dagger\right)^n e^{-iEt/\hbar}\,|\Phi_0\rangle\,, \tag{28}$$

which can be found in the literature[4].

We may go one step further which is very instructive for the sake of interpretation. Based on the expressions (13) for the variable q_1 we may define

$$\cos\theta = \frac{p}{\sqrt{2\hbar m\omega}}\ ;\ \sin\theta = \sqrt{\frac{m\omega}{2\hbar}}q \tag{29}$$

which implies

$$\tan\theta = \frac{m\omega q}{p} \tag{30}$$

that defines the angle θ as the phase difference between the movements on the q and p-axes, in the sense that the classical solutions imply

$$p = \sqrt{2mE}\cos(\omega t+\theta)\ ;\ q = \sqrt{\frac{2E}{m\omega^2}}\sin(\omega t+\theta), \tag{31}$$

and so

$$\tan\theta = \frac{m\omega q(t_0)}{p(t_0)}.$$

This gives us the value

$$q_1 = e^{i\theta} \tag{32}$$

and (26) may be written as

$$\Phi_n(q_1,t) = e^{in\theta}e^{-iEt/\hbar}\Phi_0. \tag{33}$$

In terms of the operators a and $a^\dagger$ we get

$$a^\dagger\,|\Phi_n(q_1)\rangle = e^{i(n+1)\theta}e^{-iEt/\hbar}\,|\Phi_0\rangle\ ;\ a\,|\Phi_n(q_1)\rangle = e^{i(n-1)\theta}e^{-iEt/\hbar}\,|\Phi_0\rangle \tag{34}$$

apart from normalizations. These equations fix the interpretation of the operators a and $a^\dagger$. These operators act as excitation or deexcitation of the normal modes defined upon phase space for the harmonic oscillator. Transforming to these normal modes signify that we go from our phase space ellipse defined by the hamiltonian (1) into an hyperbole defined by the hamiltonian (14).

3 Half-Integral Spin Particles

We now pass to the study of particles with half-integral spin. We will base this study on the equations previously obtained by ourselves[5, 6] for such particles.

The functions involved in this study were

$$S_1 = \frac{1}{2\omega}\left(\frac{1}{m}p_x p_y + m\omega^2 xy\right) \;;\; S_2 = \frac{1}{4\omega}\left[m\omega^2\left(x^2 - y^2\right) + \frac{1}{m}\left(p_x^2 - p_y^2\right)\right]$$

$$S_3 = \frac{1}{2}\left(xp_y - yp_x\right) \;;\; S_0 = \frac{1}{2\omega}\left[\frac{1}{m}\left(p_x^2 + p_y^2\right) + m\omega^2\left(x^2 + y^2\right)\right] \tag{35}$$

and

$$S'^2 = \frac{1}{4}S_0^2, \tag{36}$$

where we have put

$$\sqrt{\frac{\alpha}{\beta}} = m\omega. \tag{37}$$

The problem consists in making diagonal the operator related with function S'^2 and the operator related with one of the functions S_i. To make the operator S'^2 diagonal is the same as to make diagonal the operator $\widehat{S}_0$ because of (36). Indeed, we have shown[6] that if

$$\widehat{S}_0\psi = \hbar\lambda\psi, \tag{38}$$

then

$$\widehat{S}^2\psi = \hbar^2\left(\frac{\lambda - 1}{2}\right)\left(\frac{\lambda + 1}{2}\right)\psi, \tag{39}$$

where we have used

$$\widehat{S}^2 = \widehat{S}'^2 - \hbar^2/4 \tag{40}$$

or, if we put

$$\widehat{S}^2\psi = \hbar^2\left(\frac{N}{2}\right)\left(\frac{N}{2} + 1\right)\psi \tag{41}$$

then

$$N = \lambda - 1. \tag{42}$$

We now introduce the following canonical transformation

$$q_1 = \left[\frac{p_x}{\sqrt{2\hbar m\omega}} + i\sqrt{\frac{m\omega}{2\hbar}}x\right] \;;\; p_1 = \left[\frac{p_x}{\sqrt{2\hbar m\omega}} - i\sqrt{\frac{m\omega}{2\hbar}}x\right] \tag{43}$$

and

$$q_2 = \left[\frac{p_y}{\sqrt{2\hbar m\omega}} + i\sqrt{\frac{m\omega}{2\hbar}}y\right] \;;\; p_2 = \left[\frac{p_y}{\sqrt{2\hbar m\omega}} - i\sqrt{\frac{m\omega}{2\hbar}}y\right] \tag{44}$$

to write

$$S_0' = \hbar\left(q_1 p_1 + q_2 p_2\right) \tag{45}$$

where the prime indicates that the functions is written in the transformed coordinate system.

The functions S_i are, after the canonical transformation,

$$S_1' = \frac{\hbar}{2i}\left[\left(q_1^2 - q_2^2\right) + \left(p_1^2 - p_2^2\right)\right] \; ; \; S_2' = \frac{\hbar}{2}\left(q_1 p_1 - q_2 p_2\right) \tag{46}$$

and

$$S_3' = \frac{\hbar}{2i}\left(q_1 p_2 - q_2 p_1\right). \tag{47}$$

We then choose to make diagonal both S_0' and S_2'

$$S_2' = \frac{\hbar}{2}\left(q_1 p_1 - q_2 p_2\right) \; ; \; S_0' = \hbar\left(q_1 p_1 + q_2 p_2\right). \tag{48}$$

When making the quantization defined by the application of the Infinitesimal Wigner-Moyal Transformation (as above) we find the two operators

$$\widehat{S}_2' = \frac{\hbar}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) \; ; \; \widehat{S}_0' = \hbar\left(a_1^\dagger a_1 + a_2^\dagger a_2 + 1\right) \tag{49}$$

with the conventional definitions for the second quantization operators

$$a_1^\dagger = q_1 \; ; \; a_1 = -i\hbar\frac{\partial}{\partial q_1} \; ; \; a_2^\dagger = q_2 \; ; \; a_2 = -i\hbar\frac{\partial}{\partial q_2}. \tag{50}$$

In spite of working with the operator $\widehat{S}_0'$ we may, as in expression (40), work with the operator

$$\widehat{N} = \hbar\left(a_1^\dagger a_1 + a_2^\dagger a_2\right) \tag{51}$$

with eigenvalue

$$N = \lambda - 1. \tag{52}$$

In terms of the total 'angular momentum' we have the similar problem to make diagonal the operators

$$\widehat{S}_2' = \frac{\hbar}{2}\left(a_1^\dagger a_1 - a_2^\dagger a_2\right) \; ; \; \widehat{S}'^2 = \hbar\left(\frac{\widehat{N}}{2}\right)\left(\frac{\widehat{N}}{2} + 1\right). \tag{53}$$

The related eigenvectors are given by the tensor product

$$\left|\Phi_{n_1}\Phi_{n_2}\right\rangle = (a_1^\dagger)^{n1}(a_2^\dagger)^{n2}e^{-iEt/\hbar}\left|\Phi_0^1\Phi_0^2\right\rangle \tag{54}$$

which are precisely the (not normalized) eigenvectors found in the literature [7, 8]. The same considerations about the relative phase in phase space may be approached with the same methods which yield the same interpretation for the operators a_i and $a_i^\dagger$, $i = 1, 2$.

4 Conclusions

We have thus shown how to go from the formalism of classical mechanics to the second quantization one—both for integral and half-integral spin particles.

As was already said, this paper serves for a double intention. The first one is completeness. We have taken the task of showing that *all* quantum mechanical formalism may be derived from the classical one by means of the infinitesimal Wigner-Moyal Transformation. This was done.

The second intention is much more ambitious. As everyone knows, second quantization is the touchstone of quantum field theory. We hope that the study of second quantization within the interpretation here proposed may help one to understand the infinities occurring in the formalism of quantum field theory and so gives us the means to develop a more acceptable theory. This second aim will be dealt with in a future paper.

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